

MEASURING CONTRADICTION BETWEEN TWO AIFS*

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This paper is devoted to introduce an axiomatic model to distinguish what functions are suitable for measuring the degree of contradiction between two Atanassov's intuitionistic fuzzy sets. After stating the needed background, in section 2, we justify and present the axioms that a contradiction measure must satisfy, and the first examples are set out. After motivating the necessity of achieving some definition for modelling the continuity, in the next section we introduce the concepts of semicontinuity from below and semicontinuity from above for contradiction measures. Finally, in section 4, some families of contradiction measures are constructed.

Keywords: Atanassov's Intuitionistic fuzzy sets; contradiction measures; continuity from below and from above.

1. Preliminaries

As it is well known, an *Atanassov's intuitionistic fuzzy set* (AIFS) on a universe X is a set $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$, where $\mu_A, \nu_A : X \rightarrow [0, 1]$ satisfy that $\mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$ (see¹). Hence, A is an \mathbb{L} -fuzzy set (see²), where $\mathbb{L} = \{\alpha = (\alpha_1, \alpha_2) \in [0, 1]^2 \mid \alpha_1 + \alpha_2 \leq 1\}$ is a bounded and complete lattice with the order defined by: $(\alpha_1, \alpha_2) \leq_{\mathbb{L}} (\beta_1, \beta_2)$ if and only if $\alpha_1 \leq \beta_1$ and $\alpha_2 \geq \beta_2$; and with lowest element, $0_{\mathbb{L}} = (0, 1)$, and greatest element, $1_{\mathbb{L}} = (1, 0)$. Thus $\chi^A = (\mu_A, \nu_A) \in \mathbb{L}^X$ is the \mathbb{L} -membership function of A . Moreover, we say that A is \mathbb{L} -normal if $\inf_{x \in X} (\nu_A(x)) = 0$ holds, and we denote $\mathbb{L}_0^X = \{\chi \in \mathbb{L}^X \mid \chi \text{ is } \mathbb{L}\text{-normal}\}$.

Similarly to the fuzzy case, if \mathcal{N} is a *strong intuitionistic fuzzy negation* (S-IFN)(that is, $\mathcal{N} : \mathbb{L} \rightarrow \mathbb{L}$ is a decreasing function with $\mathcal{N}(0_{\mathbb{L}}) = 1_{\mathbb{L}}$,

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$\mathcal{N}(1_{\mathbb{L}}) = 0_{\mathbb{L}}$ and $\mathcal{N}(\mathcal{N}(\alpha)) = \alpha$ for all $\alpha \in \mathbb{L}$; see³ or⁴), two AIFS A and B , or alternatively χ^A and χ^B , are \mathcal{N} -contradictory if $\chi^A(x) \leq_{\mathbb{L}} (\mathcal{N} \circ \chi^B)(x)$ for all $x \in X$. Also, A and B , or χ^A and χ^B , are contradictory if they are \mathcal{N} -contradictory regarding some S-IFN \mathcal{N} (see⁵).

If \mathcal{N} is an S-IFN and N the strong fuzzy negation associated with \mathcal{N} according to the representation theorem of S-IFN (that is, $\mathcal{N}(\alpha_1, \alpha_2) = (N(1 - \alpha_2), 1 - N(\alpha_1))$, for all $(\alpha_1, \alpha_2) \in \mathbb{L}$, see³ and⁴), then χ^A and χ^B are \mathcal{N} -contradictory if and only if $N(\mu_A(x)) + \nu_B(x) \geq 1$ and $N(\mu_B(x)) + \nu_A(x) \geq 1$, for all $x \in X$ (see⁵); furthermore, χ^A and χ^B are \mathcal{N} -contradictory if and only if $g(\mu_A(x)) + g(1 - \nu_B(x)) \leq 1$ and $g(\mu_B(x)) + g(1 - \nu_A(x)) \leq 1$, $\forall x \in X$, where g is the generator of N (that is, $N(\alpha) = g^{-1}(1 - g(\alpha))$ for all $\alpha \in [0, 1]$, see⁶).

In⁷, Cubillo *et al.* analyzed the regions of $[0, 1]^2$ in which \mathcal{N} -contradictory sets are located, with the purpose of suggesting some way for measuring how contradictory two AIFS are. Given $\chi^A = (\mu_A, \nu_A)$, $\chi^B = (\mu_B, \nu_B) \in \mathbb{L}^X$, if $\chi^{AB} = (\mu_A, \nu_B)$ and $\chi^{BA} = (\mu_B, \nu_A)$, then χ^A and χ^B are \mathcal{N} -contradictory if and only if $\chi^{AB}(X), \chi^{BA}(X) \subset \mathcal{R}_{\mathcal{N}} = \{(\alpha_1, \alpha_2) \in [0, 1]^2 \mid N(\alpha_1) + \alpha_2 \geq 1\}$; and $\mathcal{R}_{\mathcal{N}}$ is called the *region of \mathcal{N} -contradiction* (see Figure 1).

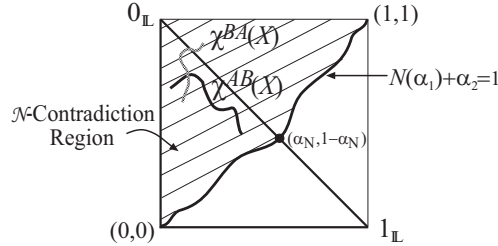


Fig. 1. Region of \mathcal{N} -contradiction and the \mathcal{N} -contradictory sets A and B .

2. Axioms for measuring contradiction between two AIFS

In⁷, some functions were proposed for measuring both the degree of \mathcal{N} -contradiction respect to a strong negation \mathcal{N} , and the degree of contradiction between two AIFS. In order to define an axiomatic model including their main features, let us see the following remarks. Firstly, as in the classical Set Theory, the empty set is included in its complementary, it means that the empty set is contradictory with itself, so the contradiction of empty AIFS with itself should be the highest, that is 1. Secondly, as it was shown in⁷, if two AIFS A and B are non-contradictory then one of them is \mathbb{L} -normal, thus the contradiction between an \mathbb{L} -normal set and any AIFS should be 0. Thirdly, the definition of \mathcal{N} -contradiction between

two AIFS is symmetrical since \mathcal{N} is an S-IFN; so a contradiction measure should be symmetrical. Finally, if A and B are two AIFS such that $\chi^A \leq_{\mathbb{L}} \chi^B$ (that is, $\chi^A(x) \leq_{\mathbb{L}} \chi^B(x)$ for all $x \in X$), then for any AIFS C that is \mathcal{N} -contradictory with B , it is $\chi^A \leq_{\mathbb{L}} \chi^B \leq_{\mathbb{L}} \mathcal{N} \circ \chi^C$, and thus “ B is closer” to be non-contradictory with C than A , which means that B is less contradictory with C than A ; therefore the anti-monotonicity is a suitable requirement. These remarks lead us to the following definition.

Definition 2.1. Let $X \neq \emptyset$, a function $\mathcal{C} : \mathbb{L}^X \times \mathbb{L}^X \rightarrow [0, 1]$ is a *measure of contradiction* on AIFS, or on \mathbb{L}^X , if it satisfies:

- (c.i) $\mathcal{C}(\chi^{0_{\mathbb{L}}}, \chi^{0_{\mathbb{L}}}) = 1$, where $\chi^{0_{\mathbb{L}}}$ is the empty AIFS: $\chi^{0_{\mathbb{L}}}(x) = 0_{\mathbb{L}} \forall x \in X$.
- (c.ii) Given $\chi^A, \chi^B \in \mathbb{L}^X$, if χ^A or χ^B is \mathbb{L} -normal, then $\mathcal{C}(\chi^A, \chi^B) = 0$.
- (c.iii) Symmetry: $\mathcal{C}(\chi^A, \chi^B) = \mathcal{C}(\chi^B, \chi^A)$ for all $\chi^A, \chi^B \in \mathbb{L}^X$.
- (c.iv) Anti-monotonicity: Given $\chi^A, \chi^B \in \mathbb{L}^X$ such that $\chi^A \leq_{\mathbb{L}} \chi^B$, then $\mathcal{C}(\chi^A, \chi^C) \geq \mathcal{C}(\chi^B, \chi^C)$, for all $\chi^C \in \mathbb{L}^X$.

The set of all contradiction measures on \mathbb{L}^X will be denoted by $\mathcal{CM}(\mathbb{L}^X \times \mathbb{L}^X)$.

In ⁷, it was proved that the previous axioms are satisfied by the following functions: Let \mathcal{N} be an S-IFN associated with N , and g the generator of N , if $\chi^A = (\mu_A, \nu_A), \chi^B = (\mu_B, \nu_B) \in \mathbb{L}^X$

$$\begin{aligned} \mathcal{C}_N(\chi^A, \chi^B) &= \text{Max}(0, \text{Min}(F_N(\mu_A, \nu_B), F_N(\mu_B, \nu_A))) \\ \mathcal{C}_g(\chi^A, \chi^B) &= \text{Max}(0, \text{Min}(F_g(\mu_A, \nu_B), F_g(\mu_B, \nu_A))) \\ \mathcal{C}_E(\chi^A, \chi^B) &= \frac{\text{Min}(d(\chi^{AB}(X), \mathcal{R}_{\mathcal{N}}^*), d(\chi^{BA}(X), \mathcal{R}_{\mathcal{N}}^*))}{d(0_{\mathbb{L}}, \mathcal{R}_{\mathcal{N}}^*)} \end{aligned} \quad (1)$$

where, for all $\mu, \nu \in [0, 1]^X$, $F_N(\mu, \nu) = \inf_{x \in X} (N(\mu(x)) + \nu(x) - 1)$, $F_g(\mu, \nu) = 1 - \sup_{x \in X} (g(\mu(x)) + g(1 - \nu(x)))$, $\mathcal{R}_{\mathcal{N}}^* = \{(\alpha_1, \alpha_2) \in [0, 1]^2 \mid N(\alpha_1) + \alpha_2 < 1\}$ and d the euclidean distance (see Figure 2).

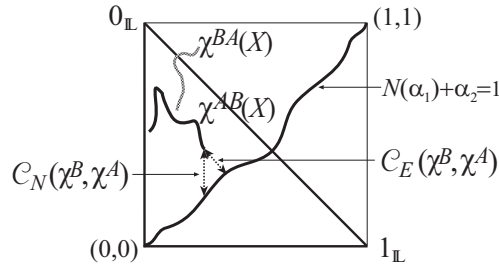


Fig. 2. Geometrical interpretation of measures \mathcal{C}_N and \mathcal{C}_E .

The previous definition does not guarantee any kind of continuity of the measures, as the following example shows: The function $\mathcal{C}_{\mathcal{N}} : \mathbb{L}^X \rightarrow [0, 1]$, given by $\mathcal{C}(\chi^A, \chi^B) = 1$ if $\chi^A = \chi^B = \chi^{0_{\mathbb{L}}}$ and $\mathcal{C}(\chi^A, \chi^B) = 0$ otherwise, is a measure of contradiction, that changes sharply in $(\chi^{0_{\mathbb{L}}}, \chi^{0_{\mathbb{L}}})$. So, if we want to modelize the continuity, we need to impose some additional conditions.

3. Axioms for modelling the contradiction measure continuity

In order to demand a measure to change smoothly, we propose new axioms.

Recall that $S \subset \mathbb{L}^X$ is a semilattice from below if for all $\chi^A, \chi^B \in S$, $\text{Sup}\{\chi^A, \chi^B\} \in S$ is satisfied; and similarly, $S \subset \mathbb{L}^X$ is a semilattice from above if for all $\chi^A, \chi^B \in S$, $\text{Inf}\{\chi^A, \chi^B\} \in S$ is satisfied (see ⁸ or ⁹).

Definition 3.1. Let $X \neq \emptyset$ and $\mathcal{C} \in \mathcal{CM}(\mathbb{L}^X \times \mathbb{L}^X)$, then

(a) \mathcal{C} is said to be *semi-continuous from below* if it satisfies:

(c.v) For all semilattice from below $\{\chi^i\}_{i \in \mathcal{I}} \subset \mathbb{L}^X$ and for all $\chi \in \mathbb{L}^X$, the equality $\text{Inf}_{i \in \mathcal{I}} \mathcal{C}(\chi^i, \chi) = \mathcal{C}(\text{Sup}_{i \in \mathcal{I}} \chi^i, \chi)$ holds, where $\text{Sup}_{i \in \mathcal{I}} \chi^i \in \mathbb{L}^X$ is defined as

$$(\text{Sup}_{i \in \mathcal{I}} \chi^i)(x) = \text{Sup}_{i \in \mathcal{I}} \chi^i(x), \text{ for all } x \in X.$$

(b) \mathcal{C} is said to be *semi-continuous from above* if it satisfies:

(c.vi) For all semilattice from above $\{\chi^i\}_{i \in \mathcal{I}} \subset \mathbb{L}^X \setminus \mathbb{L}_0^X$ and for all $\chi \in \mathbb{L}^X \setminus \mathbb{L}_0^X$, the equality $\text{Sup}_{i \in \mathcal{I}} \mathcal{C}(\chi^i, \chi) = \mathcal{C}(\text{Inf}_{i \in \mathcal{I}} \chi^i, \chi)$ holds, where $\text{Inf}_{i \in \mathcal{I}} \chi^i \in \mathbb{L}^X$

is defined as $(\text{Inf}_{i \in \mathcal{I}} \chi^i)(x) = \text{Inf}_{i \in \mathcal{I}} \chi^i(x)$ for all $x \in X$.

Remark 3.1. Notice that, in section (b) of the previous definition, it is necessary to consider the AIFS be non-L-normal. Indeed, let X be an infinite universe and let $\mathcal{P}_F(X)$ be the family of all finite subsets of X . We consider the semilattice from above $\{\chi^A\}_{A \in \mathcal{P}_F(X)} \subset \mathbb{L}^X$ such that for each $A \in \mathcal{P}_F(X)$, $\chi^A(x) = 0_{\mathbb{L}}$ if $x \in A$, and $\chi^A(x) = 1_{\mathbb{L}}$ if $x \notin A$. Thus, $\text{Inf}_{A \in \mathcal{P}_F(X)} \chi^A = \chi^{0_{\mathbb{L}}}$ and, if $\mathcal{C} \in \mathcal{CM}(\mathbb{L}^X \times \mathbb{L}^X)$, it is $\mathcal{C}(\text{Inf}_{A \in \mathcal{P}_F(X)} \chi^A, \chi^{0_{\mathbb{L}}}) = 1$, nevertheless, as $\chi^A \in \mathbb{L}_0^X$ for all $A \in \mathcal{P}_F(X)$, $\text{Sup}_{A \in \mathcal{P}_F(X)} \mathcal{C}(\chi^A, \chi^{0_{\mathbb{L}}}) = 0$.

The set of all semi-continuous from below contradiction measures will be denoted by $\mathcal{CM}_{sc}(\mathbb{L}^X \times \mathbb{L}^X)$, and the set of all semi-continuous from above contradiction measures by $\mathcal{CM}^{sc}(\mathbb{L}^X \times \mathbb{L}^X)$.

It is easy to prove that both (c.v) and (c.vi) imply (c.iv).

4. Some families of contradiction measures

In this section, we present some contradiction measures satisfying some of the axioms of continuity; proofs will be omitted due to limits of space.

Proposition 4.1. For any strong fuzzy negation N and for any order automorphism of unit interval g , functions $\mathcal{C}_N, \mathcal{C}_g$ and \mathcal{C}_E defined in equation (1) satisfy that $\mathcal{C}_N, \mathcal{C}_g, \mathcal{C}_E \in \mathcal{CM}_{sc}(\mathbb{L}^X \times \mathbb{L}^X)$ but $\mathcal{C}_N, \mathcal{C}_g, \mathcal{C}_E \notin \mathcal{CM}^{sc}(\mathbb{L}^X \times \mathbb{L}^X)$.

Proposition 4.2. Given $p > 0$, for each $\beta \in [0, 1]$ let us consider the segment $L_\beta = \left\{ (\alpha_1, \alpha_2) \in [0, 1]^2 \mid \alpha_1 \in [0, 1], \alpha_2 = \frac{(\alpha_1 + p)(1 - \beta)}{\beta + p} \right\}$. Let $\mathcal{C}_p^L, \mathcal{C}_p^U : \mathbb{L}^X \times \mathbb{L}^X \rightarrow [0, 1]$ be defined for each $\chi^A = (\mu_A, \nu_A), \chi^B = (\mu_B, \nu_B) \in \mathbb{L}^X$ by

$$\mathcal{C}_p^L(\chi^A, \chi^B) = \begin{cases} 0, & \text{if } \chi^A \in \mathbb{L}_0^X \text{ or } \chi^B \in \mathbb{L}_0^X \\ \text{Min}(1 - \beta, 1 - \beta'), & \text{if } \text{Sup}_{x \in X} \chi^{AB}(x) \in L_\beta \text{ \& } \text{Sup}_{x \in X} \chi^{BA}(x) \in L_{\beta'} \end{cases}$$

$$\mathcal{C}_p^U(\chi^A, \chi^B) = \begin{cases} 0, & \text{if } \chi^A \in \mathbb{L}_0^X \text{ or } \chi^B \in \mathbb{L}_0^X \\ \text{Max}(1 - \beta, 1 - \beta'), & \text{if } \text{Inf}_{x \in X} \chi^{AB}(x) \in L_\beta \text{ \& } \text{Inf}_{x \in X} \chi^{BA}(x) \in L_{\beta'} \end{cases}$$

Then, for all $p > 0$, $\mathcal{C}_p^L \in \mathcal{CM}_{sc}(\mathbb{L}^X \times \mathbb{L}^X)$ but $\mathcal{C}_p^L \notin \mathcal{CM}^{sc}(\mathbb{L}^X \times \mathbb{L}^X)$, and $\mathcal{C}_p^U \in \mathcal{CM}^{sc}(\mathbb{L}^X \times \mathbb{L}^X)$ but $\mathcal{C}_p^U \notin \mathcal{CM}_{sc}(\mathbb{L}^X \times \mathbb{L}^X)$ (see Figure 3).

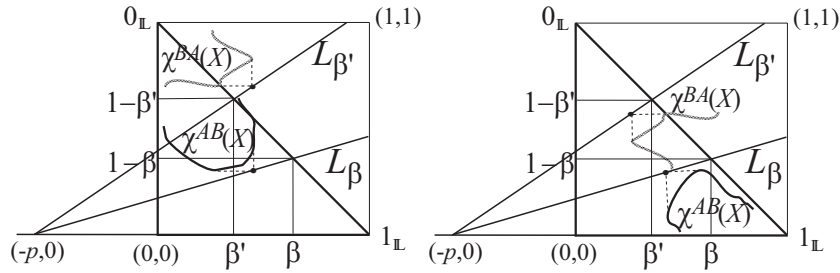


Fig. 3. Geometrical interpretation of measures \mathcal{C}_p^L and \mathcal{C}_p^U .

Proposition 4.3. For each $p > 0$, the functions $\mathcal{C}_p^\vee, \tilde{\mathcal{C}}_p : \mathbb{L}^X \times \mathbb{L}^X \rightarrow [0, 1]$, defined for each $\chi^A = (\mu_A, \nu_A), \chi^B = (\mu_B, \nu_B) \in \mathbb{L}^X$ by

$$\mathcal{C}_p^\vee(\chi^A, \chi^B) = \text{Min} \left(\frac{p \inf_{x \in X} \nu_A(x)}{p + \sup_{x \in X} \mu_B(x)}, \frac{p \inf_{x \in X} \nu_B(x)}{p + \sup_{x \in X} \mu_A(x)} \right)$$

$$\tilde{\mathcal{C}}_p(\chi^A, \chi^B) = \text{Min} \left(\inf_{x \in X} \frac{p \nu_A(x)}{p + \mu_B(x)}, \inf_{x \in X} \frac{p \nu_B(x)}{p + \mu_A(x)} \right),$$

satisfy that $\mathcal{C}_p^\vee, \tilde{\mathcal{C}}_p \in \mathcal{CM}_{sc}(\mathbb{L}^X \times \mathbb{L}^X)$, but $\mathcal{C}_p^\vee, \tilde{\mathcal{C}}_p \notin \mathcal{CM}^{sc}(\mathbb{L}^X \times \mathbb{L}^X)$.

Proposition 4.4. For each $p > 0$, the functions $\mathcal{C}_p^\wedge, \tilde{\mathcal{C}}_p : \mathbb{L}^X \times \mathbb{L}^X \rightarrow [0, 1]$, defined for each $\chi^A = (\mu_A, \nu_A), \chi^B = (\mu_B, \nu_B) \in \mathbb{L}^X$ by

$$\mathcal{C}_p^\wedge(\chi^A, \chi^B) = \begin{cases} 0, & \text{if } \chi^A \in \mathbb{L}_0^X \text{ or } \chi^B \in \mathbb{L}_0^X \\ \text{Max} \left(\frac{p \sup_{x \in X} \nu_A(x)}{p + \inf_{x \in X} \mu_B(x)}, \frac{p \sup_{x \in X} \nu_B(x)}{p + \inf_{x \in X} \mu_A(x)} \right), & \text{if } \chi^A, \chi^B \in \mathbb{L}^X \setminus \mathbb{L}_0^X \end{cases}$$

$$\tilde{\mathcal{C}}_p(\chi^A, \chi^B) = \begin{cases} 0, & \text{if } \chi^A \in \mathbb{L}_0^X \text{ or } \chi^B \in \mathbb{L}_0^X \\ \text{Max} \left(\sup_{x \in X} \frac{p\nu_A(x)}{p + \mu_B(x)}, \sup_{x \in X} \frac{p\nu_B(x)}{p + \mu_A(x)} \right), & \text{if } \chi^A, \chi^B \in \mathbb{L}^X \setminus \mathbb{L}_0^X, \end{cases}$$

satisfy that $\mathcal{C}_p^\wedge, \tilde{\mathcal{C}}_p \in \mathcal{CM}^{sc}(\mathbb{L}^X \times \mathbb{L}^X)$, but $\mathcal{C}_p^\wedge, \tilde{\mathcal{C}}_p \notin \mathcal{CM}_{sc}(\mathbb{L}^X \times \mathbb{L}^X)$.

5. Conclusions

In the framework of AIFS, contradiction is a fact that must be considered and controlled, being necessary to measure how contradictory two sets are, in order to limit the degree accepted in any specific problem. This paper proposes a general model establishing the minimum requirements a function must satisfy for suitably measuring contradiction between two sets. Axioms presented take in account different kinds of continuity. Furthermore some families of functions are studied in accordance with this point of view.

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